# Inequalities

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## The Arithmetic Mean – Geometric Mean (AM-GM) Inequality (more than two variables):

Suppose we have $n$ positive real numbers $x_1, x_2, \ldots, x_n$. Then

$$\frac{x_1 + x_2 + \cdots + x_n}{n} \geq (x_1 x_2 \cdots x_n)^{\frac{1}{n}}$$

with equality if and only if all of the numbers $x_1, x_2, \ldots, x_n$ are equal.

**Problem 1.** (Dublin Are Selection Test 2016)

Prove that for any positive real numbers $a, b$ and $c$ we have

$$\frac{2a+b}{b+2c} + \frac{2b+c}{c+2a} + \frac{2c+a}{a+2b} \geq 3.$$

**Solution.** Let

$$b + 2c = x \qquad (1)$$

$$c + 2a = y \qquad (2)$$

$$a + 2b = z \qquad (3)$$

Adding the above equalities we find

$$2a + 2b + 2c = \frac{2(x+y+z)}{3} \qquad (4)$$

Now, from (1) and (4) we find

$$2a + b = \frac{2y + 2z - x}{3}$$

and similarly,

$$2b + c = \frac{2x + 2z - y}{3} \qquad \text{and} \qquad 2c + a = \frac{2x + 2y - z}{3}.$$

Thus, in the new variables $x, y, z$ our initial inequality reads

$$\frac{1}{3}\left\{\frac{2y + 2z - x}{x} + \frac{2x + 2z - y}{y} + \frac{2x + 2y - z}{z}\right\} \geq 3,$$

or even

$$2\left(\frac{x}{y} + \frac{y}{x}\right) + 2\left(\frac{y}{z} + \frac{z}{y}\right) + 2\left(\frac{x}{z} + \frac{z}{x}\right) \geq 12. \qquad (5)$$

By AM-GM inequality we have

$$\frac{x}{y} + \frac{y}{x} \geq 2, \quad \frac{y}{z} + \frac{z}{y} \geq 2, \quad \frac{x}{z} + \frac{z}{x} \geq 2.$$

Adding the above inequalities we find (5) which proves our initial inequality.

**Problem 2.** (Dublin Area Selection Test 2014)

Prove that if $a$ and $b$ are positive real numbers,

$$\sqrt[3]{\frac{a}{b}}+\sqrt[3]{\frac{b}{a}}\leq\sqrt[3]{2\,(a+b)\left(\frac{1}{a}+\frac{1}{b}\right)}.$$

**Solution:**

Cubing both sides yields

$$\frac{a}{b}+3\left(\sqrt[3]{\frac{a}{b}}\right)^2\left(\sqrt[3]{\frac{b}{a}}\right)+3\left(\sqrt[3]{\frac{b}{a}}\right)^2\left(\sqrt[3]{\frac{a}{b}}\right)+\frac{b}{a}\leq 2\left(2+\frac{a}{b}+\frac{b}{a}\right).$$

Simplifying this yields

$$3\sqrt[3]{\frac{a}{b}}+3\sqrt[3]{\frac{b}{a}}\leq 4+\frac{a}{b}+\frac{b}{a}\,. \tag{1}$$

Now by the AM-GM inequality,

$$1+1+\frac{a}{b}\geq 3\sqrt[3]{\frac{a}{b}}$$

and

$$1+1+\frac{b}{a}\geq 3\sqrt[3]{\frac{b}{a}}$$

with equality in both cases if and only if $a=b$. Adding these two inequalities together yields the required inequality (1).

## The Cauchy-Schwarz Inequality:

For any real numbers

$$a_1, a_2, \ldots, a_n \quad \text{and} \quad b_1, b_2, \ldots, b_n$$

we have

$$(a_1^2 + a_2^2 + \cdots + a_n^2)(b_1^2 + b_2^2 + \cdots + b_n^2) \geq (a_1b_1 + a_2b_2 + \ldots a_nb_n)^2$$

with equality if and only if $\dfrac{a_1}{b_1} = \dfrac{a_2}{b_2} = \cdots = \dfrac{a_n}{b_n}$.

**Proof.** Consider the quantity

$$F(x) = (a_1x - b_1)^2 + (a_2x - b_2)^2 + \cdots + (a_nx - b_n)^2 \geq 0 \quad \text{for all } x \in \mathbb{R}.$$

Expanding the brackets we have

$$F(x) = (a_1^2 + a_2^2 + \cdots + a_n^2)x^2 - 2(a_1b_2 + a_2b_2 + \ldots a_nb_n)x + (b_1^2 + b_2^2 + \cdots + b_n^2),$$

that is,

$$F(x) = Ax^2 - 2Bx + C \geq 0 \quad \text{for all } x \in \mathbb{R},$$

where

$$A = a_1^2 + a_2^2 + \cdots + a_n^2,$$

$$B = a_1b_2 + a_2b_2 + \ldots a_nb_n,$$

$$C = b_1^2 + b_2^2 + \cdots + b_n^2.$$

This implies that $(2B)^2 - 4AC \leq 0$ which yields $AC \geq B^2$. Hence

$$(a_1^2 + a_2^2 + \cdots + a_n^2)(b_1^2 + b_2^2 + \cdots + b_n^2) \geq (a_1b_2 + a_2b_2 + \ldots a_nb_n)^2.$$

The equality holds when there exists $x \in \mathbb{R}$ such that $F(x) = 0$ so

$$a_1x - b_1 = a_2x - b_2 = \cdots = a_nx - b_n = 0,$$

which implies $x = \dfrac{a_1}{b_1} = \dfrac{a_2}{b_2} = \cdots = \dfrac{a_n}{b_n}$.

**Problem 3.** Prove that for any real numbers $a_1, a_2, \dots, a_n$ we have

$$n(a_1^2 + a_2^2 + \cdots + a_n^2) \geq (a_1 a_2 + \dots a_n)^2.$$

**Solution:** Apply Cauchy-Schwarz inequality with $b_1 = b_2 = \cdots = b_n = 1$.

**Problem 4.** (Dublin Area Selection Test 2015)

Let $x, y, z, w > 0$ and suppose that $xyzw = 16$. Show that

$$\frac{x^2}{x+y} + \frac{y^2}{y+z} + \frac{z^2}{z+w} + \frac{w^2}{w+x} \geq 4$$

with equality only when $x = y = z = w = 2$.

**Solution:** The Cauchy inequality gives

$$\left(\left(\frac{x}{\sqrt{x+y}}\right)^2 + \left(\frac{y}{\sqrt{y+z}}\right)^2 + \left(\frac{z}{\sqrt{z+w}}\right)^2 + \left(\frac{w}{\sqrt{w+x}}\right)^2\right) \times$$

$$((\sqrt{x+y})^2 + (\sqrt{y+z})^2 + (\sqrt{z+w})^2 + (\sqrt{w+x})^2)$$

$$\geq (x+y+z+w)^2,$$

with equality only when $x = y = z = w$. This simplifies to:

$$\left(\frac{x^2}{x+y} + \frac{y^2}{y+z} + \frac{z^2}{z+w} + \frac{w^2}{w+x}\right) \cdot 2(x+y+z+w) \geq (x+y+z+w)^2$$

and hence

$$\left(\frac{x^2}{x+y} + \frac{y^2}{y+z} + \frac{z^2}{z+w} + \frac{w^2}{w+x}\right) \geq \frac{x+y+z+w}{2}.$$

Applying the AM-GM to the right-hand term gives

$$\left(\frac{x^2}{x+y} + \frac{y^2}{y+z} + \frac{z^2}{z+w} + \frac{w^2}{w+x}\right) \geq 2\sqrt[4]{xyzw}$$

with equality only when $x = y = z = w$. Since $xyzw = 16$, the result follows at once.

**Problem 5.** Prove that for any real numbers $a_1, a_2, \dots, a_n$ and for any positive numbers $x_1, x_2, \dots, x_n$ we have

$$\frac{a_1^2}{x_1} + \frac{a_2^2}{x_2} + \cdots + \frac{a_n^2}{x_n} \geq \frac{(a_1 + a_2 + \cdots + a_n)^2}{x_1 + x_2 + \cdots + x_n}.$$

**Solution:** Apply Cauchy-Schwarz inequality for $a_1, a_2, \dots, a_n$ and $b_1 = \sqrt{x_1}, b_2 = \sqrt{x_2}, \dots, b_n = \sqrt{x_n}$.

**Note.** We could also use Induction Principle over the number $n \geq 2$ to prove this result.

## Problem 6. (IMO 1995)

Let $a, b, c$ be three postive numbers such that $abc = 1$.

Prove that

$$\frac{1}{a^3(b+c)} + \frac{1}{b^3(c+a)} + \frac{1}{c^3(a+b)} \geq \frac{3}{2}.$$

**Solution:** Denote $x = \frac{1}{a}$, $y = \frac{1}{b}$ and $z = \frac{1}{c}$.

Since $abc = 1$ we have $xyz = 1$ and our inequality to prove becomes

$$\frac{1}{\frac{1}{x^3}\left(\frac{1}{y} + \frac{1}{z}\right)} + \frac{1}{\frac{1}{y^3}\left(\frac{1}{z} + \frac{1}{x}\right)} + \frac{1}{\frac{1}{z^3}\left(\frac{1}{x} + \frac{1}{y}\right)} \geq \frac{3}{2},$$

That is (because $xyz = 1$)

$$\frac{x^2}{y+z} + \frac{y^2}{z+x} + \frac{z^2}{x+y} \geq \frac{3}{2}. \qquad (1)$$

Apply Cauchy-Schwarz inequality for

$$a_1 = \frac{x}{\sqrt{y+z}}, \quad a_2 = \frac{y}{\sqrt{z+x}}, \quad a_3 = \frac{z}{\sqrt{x+y}}$$
$$b_1 = \sqrt{y+z}, \quad b_2 = \sqrt{z+x}, \quad z = \sqrt{x+y}.$$

Thus,

$$(a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2) \geq (a_1 b_1 + a_2 b_2 + a_3 b_3)^2$$

becomes

$$\left(\frac{x^2}{y+z} + \frac{y^2}{z+x} + \frac{z^2}{x+y}\right) \cdot 2(x+y+z) \geq 3(x+y+z).$$

which simplifies to (1).

## Problem 7. (Iran Math Olympiad 1998)

Let $x, y, z > 1$ be such that $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 2$.

Prove that

$$\sqrt{x+y+z} \geq \sqrt{x-1} + \sqrt{y-1} + \sqrt{z-1}.$$

**Solution:** Note that $\dfrac{x-1}{x} + \dfrac{y-1}{y} + \dfrac{z-1}{z} = 1.$

Apply Cauchy-Schwarz inequality for

$$a_1 = \sqrt{\frac{x-1}{x}}, \quad a_2 = \sqrt{\frac{y-1}{y}}, \quad a_3 = \sqrt{\frac{z-1}{z}}$$

$$b_1 = \sqrt{x}, \quad b_2 = \sqrt{y}, \quad b_3 = \sqrt{z}.$$

Hence

$$(x+y+z)\left(\frac{x-1}{x}+\frac{y-1}{y}+\frac{z-1}{z}\right) \geq \left(\sqrt{x-1}+\sqrt{y-1}+\sqrt{z-1}\right)^2,$$

so

$$\sqrt{x+y+z} \geq \sqrt{x-1} + \sqrt{y-1} + \sqrt{z-1}.$$

## Problem 8.

Let $a, b, c > 0$ be such that

$$\frac{1}{a+b+1} + \frac{1}{b+c+1} + \frac{1}{c+a+1} \geq 1.$$

Prove that

$$a+b+c \geq ab+bc+ca.$$

**Solution:** Apply Cauchy-Schwarz inequality for

$$a_1 = \sqrt{a}, \quad a_2 = \sqrt{b}, \quad a_3 = 1$$

$$b_1 = \sqrt{a}, \quad b_2 = \sqrt{c}, \quad b_3 = c.$$

We find

$$(a+b+1)(a+b+c^2) \geq (a+b+c)^2$$

that is,

$$\frac{1}{a+b+1} \leq \frac{a+b+c^2}{(a+b+c)^2}.$$

Similarly,

$$\frac{1}{b+c+1} \leq \frac{a^2+b+c}{(a+b+c)^2} \quad \text{and} \quad \frac{1}{c+a+1} \leq \frac{a+b^2+c}{(a+b+c)^2}.$$

Adding up these last three inequalities and using our hypothesis we find

$$1 \leq \frac{1}{a+b+1} + \frac{1}{b+c+1} + \frac{1}{c+a+1} \leq \frac{a^2+b^2+c^2+2(a+b+c)}{(a+b+c)^2},$$

so

$$a^2+b^2+c^2+2(a+b+c) \geq (a+b+c)^2,$$

which yields $a+b+c \geq ab+bc+ca$.

## Problem 9. (German Math Olympiad)

Let $n \geq 2$ and $x_1, x_2, \ldots, x_n$ be positive numbers with sum $S$. Prove that

$$\frac{x_1}{S - x_1} + \frac{x_2}{S - x_2} + \cdots + \frac{x_n}{S - x_n} \geq \frac{n}{n-1}.$$

**Solution:** Apply Cauchy-Schwarz inequality for

$$a_1 = \sqrt{\frac{x_1}{S - x_1}}, \quad a_2 = \sqrt{\frac{x_2}{S - x_2}}, \quad \ldots, \quad a_n = \sqrt{\frac{x_n}{S - x_n}}$$

$$b_1 = \sqrt{x_1(S - x_1)}, \quad b_2 = \sqrt{x_2(S - x_2)}, \quad \ldots, \quad b_n = \sqrt{x_n(S - x_n)}.$$

Note that

$$\begin{aligned} a_1^2 + a_2^2 + \cdots + a_n^2 &= \frac{x_1}{S - x_1} + \frac{x_2}{S - x_2} + \cdots + \frac{x_n}{S - x_n} \\ b_1^2 + b_2^2 + \cdots + b_n^2 &= x_1(S - x_1) + x_2(S - x_2) + \cdots + x_n(S - x_n) \\ &= S(x_1 + x_2 + \cdots + x_n) - (x_1^2 + x_2^2 + \cdots + x_n^2) \\ &= S^2 - T, \end{aligned}$$

where $T = x_1^2 + x_2^2 + \cdots + x_n^2$. Also, $a_1 b_1 + a_2 b_2 + \ldots a_n b_n = S$

Thus, by Cauchy-Schwarz inequality we find

$$(S^2 - T)\left(\frac{x_1}{S - x_1} + \frac{x_2}{S - x_2} + \cdots + \frac{x_n}{S - x_n}\right) \geq S^2$$

Hence

$$\frac{x_1}{S - x_1} + \frac{x_2}{S - x_2} + \cdots + \frac{x_n}{S - x_n} \geq \frac{S^2}{S^2 - T}.$$

It remains to prove that $\frac{S^2}{S^2 - T} \geq \frac{n}{n-1}$ or even

$$nT \geq S^2 \iff n(x_1^2 + x_2^2 + \cdots + x_n^2) \geq (x_1 + x_2 + \cdots + x_n)^2.$$

This last inequality follows again from the Cauchy-Schwarz ineq applied to

$$a_1 = x_1, \quad a_2 = x_2, \quad \ldots, \quad a_n = x_n$$

$$b_1 = b_2 = \cdots = b_n = 1.$$